



Some dendriform functors

Frédéric Chapoton

► To cite this version:

Frédéric Chapoton. Some dendriform functors. Opérades 2009, 2009, Luminy, France. pp.35-60. hal-00416775

HAL Id: hal-00416775

<https://hal.science/hal-00416775>

Submitted on 15 Sep 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some dendriform functors

F. Chapoton

September 15, 2009

Abstract

We make a first step towards categorification of the dendriform operad, using categories of modules over the Tamari lattices. This means that we describe some functors that correspond to part of the operad structure.

1 Introduction

The notion of dendriform algebra is one of several new kinds of algebras introduced by Jean-Louis Loday around 2000. In brief, a dendriform algebra is an associative algebra, together with a decomposition of the associative product as the sum of two binary products, with some appropriate axioms. Loday has proved that the free dendriform algebras can be described using classical combinatorial objects called planar binary trees.

Some of the most interesting properties of these algebras can be formulated in terms of the corresponding operad Dend . Loday has shown that the operad Dend is Koszul. Later, it was proved in [Cha05] that the operad Dend is anticyclic.

There has been several hints that the dendriform structures (algebras and operad) are closely related to a natural partial order on planar binary trees, called the Tamari poset. First, there is an associative product on free dendriform algebras, and the product of two planar binary trees can be described as an interval in a Tamari poset [LR02]. Next, the anticyclic structure of the operad Dend is given by a collection of matrices than can be described directly from the Tamari poset [Cha07b].

The global aim of this article would be to categorify the operad Dend . An operad is essentially a collection of Abelian groups and linear maps between them. One has to define Abelian categories and functors between them, in such a way that the Grothendieck group construction recovers the initial operad.

Such a categorification has already been obtained in [Cha08] for the Koszul dual cooperad of the dendriform operad. This dual cooperad describes dialgebras, which is another one of Loday's kinds of algebras. This was though a much simpler situation, involving only quivers of type \mathbb{A} and their products.

For the operad Dend , we will only present here some partial results. The starting idea is to use the category of modules over the Tamari poset as a categorification for the Abelian group spanned by planar binary trees. In this article, we have obtained functors corresponding to the following linear maps:

- the first composition map $\circ_1 : \text{Dend}(m) \otimes \text{Dend}(n) \rightarrow \text{Dend}(m + n - 1)$,

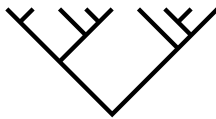


Figure 1: A planar binary tree of degree 8

- the associative product $*$: $\text{Dend}(m) \otimes \text{Dend}(n) \rightarrow \text{Dend}(m + n)$,
- a new associative product $\#$: $\text{Dend}(m) \otimes \text{Dend}(n) \rightarrow \text{Dend}(m + n - 1)$.

The product $\#$ has been explained to the author by Jean-Christophe Aval and Xavier Viennot, in the setting of Catalan alternative tableaux, see [VA] and [Vie07] for related works.

We identify the usual basis of the dendriform algebra and operad, indexed by planar binary trees, with the basis of the Grothendieck groups of the Tamari posets coming from simple modules. There is another basis of the Grothendieck groups, coming from projective modules. This will be called the basis of projective elements.

One important tool in this article is a small set-operad contained in Dend , introduced in [Cha07a]. This sub-operad can be described using noncrossing configurations in a regular polygon. We will in particular show that the basis of projective elements is contained in this sub-operad.

The reason why we have only partial results is the following : the other composition maps of the operad Dend do not preserve the set of projective elements. This makes more difficult to define the corresponding functors, even if it is possible to guess what they should be.

2 General setup

2.1 Planar binary trees

Let us first introduce very classical combinatorial objects, called **planar binary trees**. They can be concisely defined as follows: a planar binary tree is either a dot \circ or a pair of planar binary trees (xy) .

This leads to a representation as a sequence of dots and parentheses, such as

$$(((\circ\circ)(\circ(\circ\circ))) (\circ((\circ\circ)\circ))). \quad (1)$$

These objects can be converted into planar trees in a simple way. For the previous expression, the result is depicted in Figure 1. The \circ elements become leaves of the tree. Vertices of the tree correspond to pairs of matching parentheses. The outermost pair of parentheses correspond to the root vertex.

We will always draw planar binary trees with their leaves at the top, on an horizontal line, and their root at the bottom.

Let us define the degree of a planar binary tree to be the number of vertices or the number of leaves minus one. Let \mathbb{Y}_n be the set of planar binary trees of degree n . For example, $\mathbb{Y}_1 = \{\vee\}$ and $\mathbb{Y}_2 = \{\swarrow, \searrow\}$. The cardinal of \mathbb{Y}_n is the Catalan number $\frac{1}{n+1} \binom{2n}{n}$. Let \mathbb{Y} be the union of the sets \mathbb{Y}_n for $n \geq 1$.

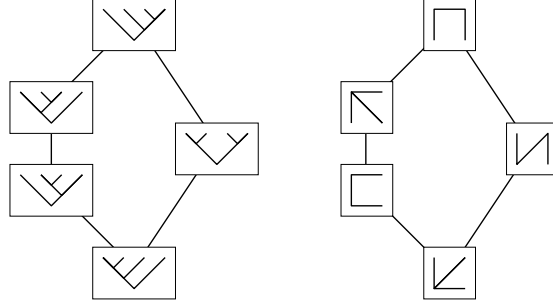


Figure 2: Left: Tamari poset Right: Good pivots between projective elements

Let us now recall two basic combinatorial operations on \mathbb{Y} : the over product $/$ and the under product \backslash , both associative and graded.

Let x, y be planar binary trees. The planar binary tree x/y is obtained by identifying the root of x with the leftmost leaf of y . Similarly $x \backslash y$ is obtained by identifying the root of y with the rightmost leaf of x .

For example, $\vee/\vee = \vee$ and $\vee\backslash\vee = \vee$.

Note that one has $(x/y)\backslash z = x/(y\backslash z)$.

Lemma 2.1 *Any planar binary tree of degree at least 2 can either be written x/y for some planar binary trees x, y or $\vee\backslash x$ for some planar binary tree x .*

Proof. If there is at least one vertex to the left of the root, then the first decomposition is possible. Else one must be in the second case. ■

There is an obvious involution on planar binary tree, the (left-right) reversal, that will be denoted $x \mapsto \bar{x}$. It exchanges the over and under products : $x/y = \bar{y}\backslash\bar{x}$

2.2 Tamari poset

There is a partial order on the set \mathbb{Y}_n , called the Tamari poset. It was introduced by Dov Tamari in [HT72] and proved there to be a lattice.

This partial order can be defined as the transitive closure of elementary moves: $x \leq y$ if x is obtained from y by a sequence of local changes, replacing the configuration \vee by the configuration \vee somewhere in the tree.

The elementary moves can be described more formally as follows.

- For any planar binary trees a, b, c , there is an elementary move from $a/\vee\backslash(b/\vee\backslash c)$ to $(a/\vee\backslash b)/\vee\backslash c$.
- If $x \rightarrow y$ is an elementary move, then $x/z \rightarrow y/z$, $z/x \rightarrow z/y$ and $\vee\backslash x \rightarrow \vee\backslash y$ are also elementary moves.

In \mathbb{Y}_n , there is a unique minimum element $\hat{0}$, which is the left comb of degree n , and a unique maximum element $\hat{1}$, which is the right comb of order n .

We will use the following convention: the Hasse diagram of the Tamari poset is drawn with its maximum element at the top and its minimum element at the

bottom. We will orient the edges of the Hasse diagram in the decreasing way (from top to bottom).

It is well-known that the Hasse diagram of the Tamari poset is the skeleton of a simple polytope, the associahedron or Stasheff polytope.

Proposition 2.2 *The Hasse diagram of the Tamari poset \mathbb{Y}_n is a regular graph of order $n - 1$.*

The reversal is an anti-automorphism of the Tamari poset: $x \leq y \Leftrightarrow \overline{y} \leq \overline{x}$.

In this article, we will consider the Hasse diagram of the Tamari poset as a quiver with relations. The arrows are the edges with the decreasing orientation. Relations are given by all possible equalities between paths. The category of representations of this quiver with relations, which is equivalent to the category of modules over the incidence algebra of the Tamari poset, will be denoted by $\text{mod } \mathbb{Y}_n$.

2.3 Dendriform operad

Let $\text{Dend}(n)$ be the free Abelian group $\mathbb{Z}\mathbb{Y}_n$. Then the collection $(\text{Dend}(n))_{n \geq 1}$ can be given the structure of an operad in the category of Abelian groups, called the Dendriform operad [Lod01].

More precisely, the collection $(\text{Dend}(n))_{n \geq 1}$ is a non-symmetric operad. This means that for all $m \geq 1, n \geq 1$ and for each $1 \leq i \leq m$, there is a linear map \circ_i from $\text{Dend}(m) \otimes \text{Dend}(n) \rightarrow \text{Dend}(m + n - 1)$. All these maps satisfy the “associativity” conditions, *i.e.* the operad axioms. Furthermore, the distinguished element \vee in $\text{Dend}(1)$ plays the role of a unit for the composition maps \circ_i .

There is a precise combinatorial description of the maps \circ_i , as some kind of double shuffle of planar binary trees. As this is not needed in the sequel, we will not recall it there.

The reversal map sends compositions to compositions:

$$\overline{x \circ_i y} = \overline{x} \circ_{m+1-i} \overline{y}, \quad (2)$$

for $x \in \text{Dend}(m)$.

2.4 Dendriform algebra

Let Dend be the direct sum of all Abelian groups $\text{Dend}(n)$ for $n \geq 1$. Then there is an associative graded product $*$ on Dend , which can be defined using the operad structure as follows:

$$x * y = ((\vee + \vee) \circ_2 y) \circ_1 x. \quad (3)$$

More concretely, by a result of Loday and Ronco in [LR02], this associative product is also given by

$$x * y = \sum_{x/y \leq z \leq x \vee y} z. \quad (4)$$

The product can also be informally described as follows: the product of two planar binary trees x and y is the sum over all planar binary trees obtained by shuffling the right side of x with the left side of y .

For instance, one has $\vee * \vee = \vee + \vee$.

Proposition 2.3 *The reversal of planar binary trees is an anti-automorphism of the $*$ product: $\overline{x * y} = \overline{y} * \overline{x}$.*

Proof. One uses definition (3), the fact that the reversal maps \circ_1 to \circ_n by (2), and the “commutativity” axiom of operads. ■

Let us note that the over and under products can also be expressed using the composition maps of the Dendriform operad:

$$x/y = (y \circ_1 \vee) \circ_1 x \quad (5)$$

and

$$x \backslash y = (x \circ_m \vee) \circ_{m+1} y, \quad (6)$$

if $x \in \mathbb{Y}_m$.

Let us give now a few useful relations.

Lemma 2.4 *One has*

$$x/(y * z) = (x/y) * z, \quad (7)$$

$$(x \circ_1 y)/z = (x/z) \circ_1 y, \quad (8)$$

$$(x \circ_1 y) * z = (x * z) \circ_1 y. \quad (9)$$

Proof. This is an exercise in the Dendriform operad, using (3), (5) and (6). ■

Let us introduce a bilinear form on $\text{Dend}(n)$. Let $x = \sum_{s \in \mathbb{Y}_n} x_s s$ and $y = \sum_{t \in \mathbb{Y}_n} y_t t$ be elements of $\text{Dend}(n)$. One defines

$$\langle x, y \rangle = \sum_{s \leq t \in \mathbb{Y}_n} x_s y_t \mu(s, t), \quad (10)$$

where μ is the Möbius function of the Tamari poset. The Möbius function μ is known to have values in $\{-1, 0, 1\}$ (see [Pal93, BW97]).

This bilinear form is called the Euler form. Note that this is not symmetric. The Euler form has a natural meaning in representation theory, namely it comes from the alternating sum of dimensions of Ext groups in the category $\text{mod } \mathbb{Y}_n$.

Let E be the associated quadratic form $E(x) = \langle x, x \rangle$.

2.5 Anticyclic structure

The operad Dend is in fact an anticyclic operad. This means that there exists, for each $n \geq 1$, a linear endomorphism τ of $\text{Dend}(n)$ satisfying $\tau(\vee) = -\vee$, $\tau^{n+1} = \text{Id}$ and the following compatibility conditions with the composition maps \circ_i of the Dendriform operad:

$$\tau(x \circ_n y) = -\tau(y) \circ_1 \tau(x), \quad (11)$$

$$\tau(x \circ_i y) = \tau(x) \circ_{i+1} y \quad \text{if } 1 \leq i < n, \quad (12)$$

where $x \in \text{Dend}(n)$ and $y \in \text{Dend}(m)$.

We have proved in [Cha07b] that there exists a linear endomorphism θ of $\text{Dend}(n)$ such that $\tau = (-1)^n \theta^2$ and with the following properties:

$$\theta(\vee) = -\vee, \quad (13)$$

$$\theta(x \setminus y) = -\theta(x) * \theta(y), \quad (14)$$

$$\theta(x * y) = -\theta(x) / \theta(y), \quad (15)$$

$$\theta^{-1}(x / y) = -\theta^{-1}(x) * \theta^{-1}(y), \quad (16)$$

$$\theta^{-1}(x * y) = -\theta^{-1}(x) \setminus \theta^{-1}(y). \quad (17)$$

The endomorphism θ is defined in [Cha07b] using only the Tamari partial order on \mathbb{Y}_n . In fact, θ has a natural meaning in representation theory: it comes from the Auslander-Reiten translation, which is an auto-equivalence of the derived category of $\text{mod } \mathbb{Y}_n$. It follows from this definition and the fact that reversal is an anti-automorphism of the Tamari poset that θ has the following property:

$$\theta(\overline{x}) = \overline{\theta^{-1}(x)}. \quad (18)$$

3 The operad of noncrossing plants

3.1 Noncrossing combinatorics

Let us now introduce other combinatorial objects: noncrossing trees and noncrossing plants.

Let $n \geq 1$. Let O_n be a convex polygon with $n + 1$ vertices, with a distinguished side called the **base side**, that we will use as bottom side. If $n \geq 2$, the other sides are numbered from 1 to n in the clockwise order. If $n = 1$, by convention, there is only one side, which is the base side and the side 1.

A **noncrossing tree** of degree n is a subset of the set of edges between vertices of the polygon O_n such that

- No two edges cross (but they can meet at their end points),
- There is no cycle,
- The collection is maximal with respect to these properties.

Let NCT_n be the set of noncrossing trees in O_n .

Remark: noncrossing trees can be identified with exceptional collections (up to permutation) in the derived category of the quiver of type A (see [Ara]).

A **noncrossing plant** of degree n is a disjoint pair of subsets of the set of edges between vertices of the polygon O_n , called numerator edges and denominator edges, such that

- No two edges cross (but they can meet at their end points),
- Each cycle of denominator edges surrounds exactly one numerator edge.
- Each numerator edge is inside a cycle of denominator edges.
- The collection is maximal with respect to these properties.

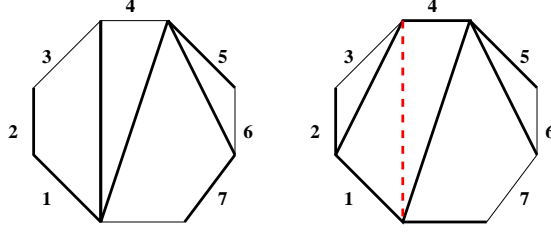


Figure 3: A noncrossing tree and a noncrossing plant in O_7

Let NCP_n be the set of noncrossing plants in O_n .

Note that noncrossing trees can be identified with noncrossing plants without numerator edges.

Remark: the numerator edges cannot be on the boundary of the polygon O_n .

An **angle** in a noncrossing tree is a pair of edges with a common endpoint v that are adjacent in the ordered set of edges incident to v .

The 3 vertices involved in an angle define a triangle. Define the base side of this triangle to be the edge which is closest to the base side of the polygon O_n . An angle is said to be of type \triangleleft , \blacktriangle or \blacktriangleright according to the noncrossing tree obtained by restriction to the triangle.

Let $N_{\triangleleft}(P)$, $N_{\blacktriangle}(P)$ and $N_{\blacktriangleright}(P)$ be the numbers of angles of type \triangleleft , \blacktriangle and \blacktriangleright in a noncrossing tree P .

Lemma 3.1 *In any noncrossing tree P of degree n , one has $n - 1$ angles: $N_{\triangleleft}(P) + N_{\blacktriangle}(P) + N_{\blacktriangleright}(P) = n - 1$.*

Proof. This is an easy combinatorial exercise. ■

3.2 Operad structure

In the article [Cha07a], it was proved that the smallest sub-operad (in the category of sets) of Dend containing the elements $\vee, \vee + \vee, \vee$ of $\text{Dend}(2)$ can be described by noncrossing plants.

The composition of noncrossing plants is given by gluing as follows. Let $P \in \text{NCP}_m$ and $Q \in \text{NCP}_n$. Let us describe $P \circ_i Q$ as a noncrossing plant in O_{m+n-1} . First consider the polygon obtained by identification of the base side of O_n with the side i of O_m . By choosing appropriate deformations of O_n and O_m , one can assume that this polygon is convex and can be identified with O_{m+n-1} , with base side the base side of O_m . Then one takes the union of P and Q inside this glued polygon, with a special care along the gluing edge. If the gluing edge belongs both to P and Q , it is kept in $P \circ_i Q$. If it belongs to just one of P or Q , then it is not kept in $P \circ_i Q$. If it does not belong to P nor to Q , then it is replaced by a numerator edge in $P \circ_i Q$.

Let us say that a noncrossing tree is **based** if it contains the base side.

Lemma 3.2 *Let P and Q be noncrossing trees. Then $P \circ_i Q$ is a noncrossing tree if and only if Q is based or P contains the border side i .*

Proof. This follows from the explicit description of the composition maps given above. The only case that should be avoided is the case when a numerator edge is created during composition. ■

The operad of noncrossing plants is generated by $\triangleleft, \blacktriangle, \blacktriangleright$ ([Cha07a, Th. 6.3]).

The inclusion of the operad of noncrossing plants in Dend is the unique morphism of operad extending the following map on generators

$$\begin{cases} \triangleleft \mapsto \searrow, \\ \blacktriangle \mapsto \searrow + \searrow, \\ \blacktriangleright \mapsto \searrow. \end{cases} \quad (19)$$

In the context of noncrossing plants, the associative $*$ product defined by (3) is given by

$$P * Q = (\blacktriangle \circ_2 Q) \circ_1 P. \quad (20)$$

This can be described as gluing P and Q on the left and right sides of a triangle \blacktriangle .

Lemma 3.3 *The set of noncrossing trees is closed under the $*$ product.*

Proof. Indeed, formula (20) shows that the product of noncrossing trees only involves compositions that do not create a numerator edge. ■

Lemma 3.4 *Each noncrossing tree has a unique decomposition as a $*$ product of based noncrossing trees.*

Proof. Let P be a noncrossing tree. There is a unique path in P from the left vertex of the base side to the right vertex of the base side. Each edge of this path can be considered as the base side of a based noncrossing tree, by restriction. Then P is the $*$ product of these noncrossing trees in their natural order. Uniqueness is clear. ■

From (5) and (6), the over product can be restated as

$$P/Q = (Q \circ_1 \triangleleft) \circ_1 P \quad (21)$$

and the under product as

$$P \setminus Q = (P \circ_m \blacktriangleright) \circ_{m+1} Q, \quad (22)$$

if $P \in \text{NCT}_m$.

One can easily see from these formulas that the set of noncrossing trees is closed with respect to the over and under products.

4 First description of projective elements

We will from now on identify the Grothendieck group of the Tamari poset \mathbb{Y}_n with the Abelian group $\text{Dend}(n) = \mathbb{Z}\mathbb{Y}_n$ by sending the simple module associated with a planar binary tree T to the same planar binary tree T in the natural basis of $\text{Dend}(n)$.

Let $P(x)$ be the projective module for the Tamari poset associated with the vertex x . More precisely, $P(x)$ is the \mathbb{Y}_n -module defined at the level of vertices by a copy of \mathbb{Q} for each element of the interval $\{y \in \mathbb{Y}_n \mid \widehat{0} \leq y \leq x\}$ and the null vector space elsewhere, and by the identity map when possible and the 0 map else.

For $x \in \mathbb{Y}_n$, let $\mathcal{P}(x)$ be the sum of all elements of the interval $\{y \in \mathbb{Y}_n \mid \widehat{0} \leq y \leq x\}$. These sums will be called projective elements.

The projective element $\mathcal{P}(x)$ is therefore the image of the projective module $P(x)$ in the Grothendieck group of the Tamari poset.

Lemma 4.1 *One has $\mathcal{P}(x/y) = \mathcal{P}(x)/\mathcal{P}(y)$.*

Proof. For every x and y , there is a simple bijection

$$\begin{cases} \{\widehat{0} \leq a \leq x\} \times \{\widehat{0} \leq b \leq y\} \simeq \{\widehat{0} \leq c \leq x/y\} \\ (a, b) \mapsto a/b. \end{cases} \quad (23)$$

The existence of this map follows from properties of elementary moves. To define its inverse, one has to check that an elementary move starting from a/b is either of the shape $a/b \rightarrow a'/b$ or of the shape $a/b \rightarrow a/b'$. ■

Lemma 4.2 *One has $\mathcal{P}(\vee \setminus x) = \vee * \mathcal{P}(x) = \blacktriangle \circ_2 \mathcal{P}(x)$.*

Proof. This follows from the description of such intervals obtained in [Cha07, Prop. 4.1]. ■

Proposition 4.3 *A projective element of degree at least 2 can either be written as P/Q for some projective elements P and Q or as $\blacktriangle \circ_2 P$ for some projective element P .*

Proof. This follows from Lemmas 2.1, 4.1 and 4.2. ■

Proposition 4.4 *If P and Q are projective elements, then P/Q is a projective element.*

Proof. If $P = \mathcal{P}(x)$, $Q = \mathcal{P}(y)$ then $P/Q = \mathcal{P}(x/y)$ by Lemma 4.1. ■

Proposition 4.5 *If P is a projective element, then $\blacktriangle \circ_2 P$ is a projective element.*

Proof. If $P = \mathcal{P}(x)$ then $\blacktriangle \circ_2 P = \mathcal{P}(\vee \setminus x)$ by Lemma 4.2. ■

Theorem 4.6 *Projective elements are noncrossing trees containing the border side 1.*

Proof. By induction on the degree n . This is clearly true if $n = 1$, for \triangleleft and \blacktriangle .

Let P be a projective element of degree $n \geq 2$. One can use Prop. 4.3.

If P can be written Q/R with Q and R projective elements, then the induction hypothesis for Q and R implies (using Lemma 3.2) that P is a noncrossing tree containing the border side 1.

Else P can be written $\blacktriangle \circ_2 Q$ with Q a projective element. Then the induction hypothesis for Q implies (using Lemma 3.2) that P is a noncrossing tree containing the border side 1. ■

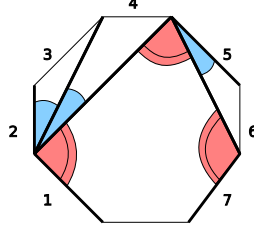


Figure 4: Angles in a projective noncrossing tree

5 Projective elements and pivots

Proposition 5.1 *In any projective element P of degree n , one has $N_{\triangleleft}(P) + N_{\blacktriangleleft}(P) = n - 1$ and $N_{\blacktriangleright}(P) = 0$.*

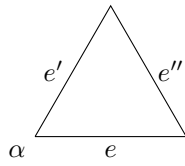
Proof. By induction on the degree n . This is clearly true if $n = 1$, for \triangleleft and \blacktriangleleft .

Let P be a projective element of degree $n \geq 2$. One can use Prop. 4.3.

If P can be written Q/R with Q and R projective elements, then, by Th. 4.6, angles in P are just angles in Q , angles in Q and a new angle of type \triangleleft . Using the induction hypothesis for Q and R , one gets the expected result.

Else P can be written $\blacktriangleleft \circ_2 Q$ with Q a projective element. In this case, by Th. 4.6, angles in P are just angles in Q and a new angle of type \blacktriangleleft . Using the induction hypothesis for Q , one gets the expected result. \blacksquare

Let us now introduce the notion of **pivot** from one noncrossing tree P to another one Q . One considers an angle of P . Let us call α the common vertex to the two adjacent edges e and e' of this angle, in trigonometric order. Let e'' be the third side of the triangle defined by this angle. Let Q be the noncrossing tree defined from P by removing e and adding e'' . This amounts to rotate clockwise the edge e .



When noncrossing trees are seen as exceptional collections (see [Ara]), a pivot corresponds to a mutation.

Definition 1 *Let P and Q be noncrossing trees. If one can go from P to Q by a pivot replacing an angle of type \triangleleft by an angle of type \blacktriangleleft , then we will say that $P \rightarrow Q$ is a good pivot.*

The following two lemmas are then quite obvious.

Lemma 5.2 *If $P \rightarrow Q$ is a good pivot, then $\blacktriangleleft \circ_2 P \rightarrow \blacktriangleleft \circ_2 Q$ is a good pivot.*

Lemma 5.3 *If $P \rightarrow Q$ is a good pivot, then $P/R \rightarrow Q/R$ is a good pivot and $R/P \rightarrow R/Q$ is a good pivot.*

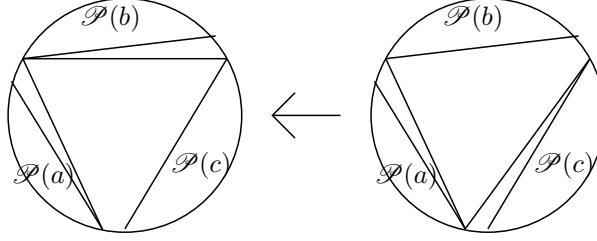


Figure 5: The good pivot corresponding to an elementary move

Lemma 5.4 *Let $x, y \in \mathbb{Y}_n$. Assume that there is an edge from y to x in the Hasse diagram of the Tamari poset. Then $\mathcal{P}(x) \rightarrow \mathcal{P}(y)$ is a good pivot.*

Proof. Edges in the Hasse diagram correspond to elementary moves of planar binary trees. Assume first that the elementary move is located at the root of a planar binary tree. In the Hasse diagram, this corresponds to an edge from $a/(\vee \setminus (b/(\vee \setminus c)))$ to $a/(\vee \setminus b)/(\vee \setminus c)$ for some planar binary trees a, b, c , possibly empty.

The corresponding projective elements are $\mathcal{P}(a)/(\vee * \mathcal{P}(b))/(\vee * \mathcal{P}(c))$ and $\mathcal{P}(a)/(\vee * (\mathcal{P}(b)/(\vee * \mathcal{P}(c))))$.

In graphical terms, one can see that these projective elements are related by a good pivot as in Fig. 5.

If the elementary move is located higher in the planar binary tree, then one has to use the description of elementary moves given in §2.2. Then one concludes by Lemmas 5.2 and 5.3. ■

Lemma 5.5 *Let P be a projective element and let Q be a noncrossing tree. If $P \rightarrow Q$ or $Q \rightarrow P$ is a good pivot, then Q is a projective element. Moreover, each good pivot between projective elements correspond to an irreducible morphism of projective modules.*

Proof. The idea is to count good pivots and irreducible morphisms between projective modules. Let $x \in \mathbb{Y}_n$. As the Hasse diagram of the Tamari poset is regular of degree $n - 1$, there are $n - 1$ edges incident to x in this Hasse diagram. But the Hasse diagram also describes irreducible morphisms between projective modules, hence there are exactly $n - 1$ irreducible morphisms from (or to) $P(x)$ to (or from) another projective module. By Lemma 5.4, each edge of the Hasse diagram is given by a good pivot between projective elements.

By Lemma 5.1, there are $n - 1$ good pivots from or to the projective element P . Hence each good pivot corresponds to an edge of the Hasse diagram and to an irreducible morphism between projective modules. ■

From now on, a good pivot from $\mathcal{P}(x)$ to $\mathcal{P}(y)$ will (slightly abusing notation) represent also the unique morphism from $P(x)$ to $P(y)$ that is the identity when possible and 0 else. From what precedes one gets the following Lemma.

Lemma 5.6 *Every two sequences of good pivots between projective modules with common start and common end give the same morphism between projective modules.*

6 Other description of projective elements

Proposition 6.1 *Any based projective element P of degree at least 2 can be written uniquely as $\triangleleft \circ_1 Q$ for some projective element Q .*

Proof. By induction on the degree n . This is clearly true for the based projective element \triangleleft . Let us use Prop. 4.3.

As P is based, it cannot be written $\blacktriangle \circ_2 R$ for some projective element R . Therefore it can be written R'/R for some projective elements R and R' . Necessarily R is based. By induction hypothesis, R can be written $\triangleleft \circ_1 R''$ for some projective element R'' . Then $Q = R'/R''$ is a projective element by Prop. 4.4 and one has $P = \triangleleft \circ_1 Q$.

Uniqueness is obvious, as there is clearly at most one noncrossing tree Q satisfying the hypothesis. ■

Proposition 6.2 *Any projective element P has a unique decomposition as a $*$ product of based projective elements.*

Proof. First, let us note that each noncrossing tree can be uniquely written as a product of based noncrossing trees by Lemma 3.4. Therefore uniqueness in the assertion is clear. It remains only to prove that all factors are projective elements, by induction on the degree of P .

If P is based, then it is its own decomposition.

If P is not based, let us use Prop. 4.3.

If P can be written $\blacktriangle \circ_2 Q$ for some projective Q , then one has $P = \vee * Q$. Using the induction hypothesis for Q , one gets the result for P .

Else P can be written Q/R for some projective elements Q and R . By induction hypothesis, $R = R_1 * R_2 * \cdots * R_k$ where R_i are some based projective elements. Then $P = (Q/R_1) * R_2 * \cdots * R_k$ by (7). By Prop. 4.4, the first factor Q/R_1 is projective. It is also based, hence one has obtained the wanted decomposition for P . ■

Proposition 6.3 *If P is a projective element, then $\triangleleft \circ_1 P$ is a based projective element.*

Proof. This follows from Prop. 4.4, because $\triangleleft \circ_1 P = P/\vee$. ■

Proposition 6.4 *If P and Q are projective elements, then $P * Q$ is a projective element.*

Proof. By induction on $\deg(P) + \deg(Q)$ and $\deg(P)$.

If $\deg(P) = 1$, then $P = \vee$ and $P * Q = \blacktriangle \circ_2 Q$ is a projective element by Prop. 4.5.

Assume that $\deg(P) \geq 2$. If P is based, then it can be written R/\vee for some projective element R , by Prop. 6.1. Then $P * Q = R/(\vee * Q)$ by (7). The product $\vee * Q$ is a projective element by the initial step of induction. Hence $P * Q$ is a projective element by Prop. 4.4.

If P is not based, then P can be written $R * R'$ for some projective elements, by Prop. 6.2. Then $P * Q = R * (R' * Q)$ and $R' * Q$ is a projective element by induction on the sum of degrees. Therefore $P * Q$ is a projective element by induction on the degree of the first factor. ■

Let us now give a useful characterization of projective elements.

Proposition 6.5 *A noncrossing tree P is a projective element if and only if $N_{\blacktriangle}(P) = 0$.*

Proof. If P is a projective element, then $N_{\blacktriangle}(P) = 0$ by Prop. 5.1.

Assume now that $N_{\blacktriangle}(P) = 0$. The proof that P is a projective element uses induction on the degree n . This is clear if $n = 1$, for \blacktriangleleft and \blacktriangleright .

If P is based, then it can be written $\blacktriangleleft \circ_1 Q$ for some noncrossing tree Q (one uses the hypothesis $N_{\blacktriangle}(P) = 0$ to show that the right side of P is empty). One then necessarily has $N_{\blacktriangle}(Q) = 0$. Hence Q is a projective element by induction. Therefore P is a projective element by Prop. 6.3.

If P is not based, it can be written as a product $R_1 * \dots * R_k$ for some based noncrossing trees R_1, \dots, R_k . Then one necessarily has $N_{\blacktriangle}(R_i) = 0$ for every i . Therefore each R_i is projective by induction. Hence P is projective by Prop. 6.4. \blacksquare

7 Composition of projective elements

Proposition 7.1 *Let P, Q be projective elements. Assume that $P \circ_i Q$ is a noncrossing tree. Then $P \circ_i Q$ is a projective element.*

Proof. One has to distinguish two cases.

If P contains the border side i , then the set of angles of $P \circ_i Q$ is in bijection with the disjoint union of the set of angles of P and the set of angles of Q . By this bijection, the type of each angle is preserved.

If P does not contain the border side i , then necessarily, by Prop. 3.2, Q is based. In this case, there is a bijection between the set of angles of $P \circ_i Q$ and the disjoint union of the set of angles of P and the set of angles of Q . By this bijection, the type of each angle is preserved, except maybe for one angle of type \blacktriangleleft of Q between the base side and the border side 1 of Q , which may give an angle of type \blacktriangleleft or \blacktriangleright in $P \circ_i Q$.

In both cases, one therefore has $N_{\blacktriangle}(P \circ_i Q) = 0$. By Prop. 6.5, one gets the result. \blacksquare

Corollary 7.2 *Let P, Q be projective elements. Then $P \circ_1 Q$ is a projective element.*

Proof. This follows from Proposition 7.1 and Th. 4.6. \blacksquare

8 The composition functor \circ_1

Consider the following subset

$$\mathcal{M}_{m,1}^n = \{(x, y, z) \in \mathbb{Y}_m \times \mathbb{Y}_n \times \mathbb{Y}_{m+n-1} \mid z \in \mathcal{P}(x) \circ_1 \mathcal{P}(y)\}. \quad (24)$$

Let us now define a module $\mathbb{M}_{m,1}^n$ over the quiver $\mathbb{Y}_m^{op} \times \mathbb{Y}_n^{op} \times \mathbb{Y}_{m+n-1}$ with all possible commuting relations.

The module $\mathbb{M}_{m,1}^n$ is given on vertices by a copy of \mathbb{Q} at each element of $\mathcal{M}_{m,1}^n$ and the null vector space elsewhere. On the level of maps, it is given by the Id map whenever possible and the 0 map else.

One then has to check that relations are satisfied.

Proposition 8.1 *The module $\mathbb{M}_{m,1}^n$ is a module over the quiver $\mathbb{Y}_m^{op} \times \mathbb{Y}_n^{op} \times \mathbb{Y}_{m+n-1}$ with all commuting relations.*

Proof. First, let us show that $\mathbb{M}_{m,1}^n$ is a \mathbb{Y}_{m+n-1} -module.

Indeed, it decomposes (when restricted to the arrows coming from \mathbb{Y}_{m+n-1}) as a direct sum over x and y , where the component associated with x, y has support $\mathcal{P}(x) \circ_1 \mathcal{P}(y)$. By Corollary 7.2, each such component is a projective \mathbb{Y}_{m+n-1} -module. This proves that $\mathbb{M}_{m,1}^n$ is a projective \mathbb{Y}_{m+n-1} -module.

Let us then prove that $\mathbb{M}_{m,1}^n$ is a $\mathbb{Y}_m^{op} \times \mathbb{Y}_{m+n-1}$ -module.

Let $x \rightarrow x'$ be an arrow in the Hasse diagram of \mathbb{Y}_m . By Lemma 5.4, there is a good pivot $\mathcal{P}(x') \rightarrow \mathcal{P}(x)$. It follows from the graphical definition of \circ_1 on noncrossing trees that there exists a sequence of good pivots starting from $\mathcal{P}(x') \circ_1 \mathcal{P}(y)$ and ending with $\mathcal{P}(x) \circ_1 \mathcal{P}(y)$. Therefore, by Proposition 5.5, there is a morphism of \mathbb{Y}_{m+n-1} -module between the corresponding projective \mathbb{Y}_{m+n-1} -modules.

Furthermore, for any $x, x' \in \mathbb{Y}_m$, any two sequences of good pivots from $\mathcal{P}(x') \circ_1 \mathcal{P}(y)$ to $\mathcal{P}(x) \circ_1 \mathcal{P}(y)$ give the same map between projective \mathbb{Y}_{m+n-1} -modules, by Lemma 5.6.

This implies that $\mathbb{M}_{m,1}^n$ is a $\mathbb{Y}_m^{op} \times \mathbb{Y}_{m+n-1}$ -module.

Let us now prove similarly that $\mathbb{M}_{m,1}^n$ is a $\mathbb{Y}_n^{op} \times \mathbb{Y}_{m+n-1}$ module.

Let $y \rightarrow y'$ be an arrow in the Hasse diagram of \mathbb{Y}_n . By Lemma 5.4, there is a good pivot $\mathcal{P}(y') \rightarrow \mathcal{P}(y)$. It follows from the graphical definition of \circ_1 on noncrossing trees that there exists a good pivot from $\mathcal{P}(x) \circ_1 \mathcal{P}(y') \rightarrow \mathcal{P}(x) \circ_1 \mathcal{P}(y)$. Therefore there is a morphism of \mathbb{Y}_{m+n-1} -module between the corresponding projective \mathbb{Y}_{m+n-1} -modules.

Furthermore, for any $y, y' \in \mathbb{Y}_n$, any two sequences of good pivots from $\mathcal{P}(x) \circ_1 \mathcal{P}(y')$ to $\mathcal{P}(x) \circ_1 \mathcal{P}(y)$ give the same map between projective \mathbb{Y}_{m+n-1} -modules, by Lemma 5.6.

This implies that $\mathbb{M}_{m,1}^n$ is a $\mathbb{Y}_n^{op} \times \mathbb{Y}_{m+n-1}$ -module.

It remains only to prove that $\mathbb{M}_{m,1}^n$ is a $\mathbb{Y}_m^{op} \times \mathbb{Y}_n^{op}$ -module. This is again a consequence of Lemma 5.6. ■

One can therefore define a composition functor \circ_1 from the category of $\mathbb{Y}_m \times \mathbb{Y}_n$ modules to the category of \mathbb{Y}_{m+n-1} modules as the tensor product with the module $\mathbb{M}_{m,1}^n$.

By definition, the functor \circ_1 induces, at the level of the Grothendieck group of the Tamari posets, the composition \circ_1 .

One consequence is the following.

Proposition 8.2 *The map \circ_1 preserves the Euler form of the Tamari posets:*

$$E(x \circ_1 y) = E(x)E(y). \quad (25)$$

Proof. This is an automatic consequence of the existence of the functor \circ_1 , as the Euler form has a natural categorical interpretation. ■

8.1 Other composition functors

It would be desirable to define the other composition functors \circ_i , for $i > 1$. So far, we have not been able to do that directly as the tensor product with

a tri-module. The point is that the composition maps \circ_i do not preserve the set of projective elements, unless $i = 1$. This makes more difficult to prove the existence of the necessary tri-module.

There is one indirect way, though, to define these functors. This requires first to dispose of an invertible functor which categorifies θ . Such a functor is given by the Auslander-Reiten translation on the derived category of the category of \mathbb{Y}_n -modules. From the relation between τ and θ , one can then define a functor which categorifies τ . One can use the axiom (12) of an anticyclic operad as a model, to define functors that categorify \circ_i .

This gives a possible definition of functors \circ_i between derived categories. It would be much better to define them at the level of categories of modules, as the \circ_i maps are known to have good positivity properties.

In any case, it is enough to have found a functor that categorify the \circ_i product, to obtain the following result.

Proposition 8.3 *For any i , the map \circ_i preserves the Euler form of the Tamari posets:*

$$E(x \circ_i y) = E(x)E(y). \quad (26)$$

This Proposition can also be deduced directly from Prop. 8.2, using the axioms of an anticyclic operad, and the relation between τ and θ .

8.2 Categorification of the $*$ product

By the same kind of argument as for \circ_1 , using Prop. 6.4 instead of Corollary 7.2, one can define a functor $*$ from the category of $\mathbb{Y}_m \times \mathbb{Y}_n$ modules to the category of \mathbb{Y}_{m+n} modules, that is a categorification of the $*$ product.

This implies the following result.

Proposition 8.4 *The $*$ product respects the Euler form of the Tamari posets: one has $E(x * y) = E(x) * E(y)$.*

9 Planar binary trees as noncrossing trees

Recall that each noncrossing tree is a sum of planar binary trees without multiplicity. Let us now characterize when this sum has only one term.

Lemma 9.1 *A noncrossing tree P is a single planar binary tree if and only if $N_{\blacktriangle}(P) = 0$. Moreover, this defines a bijection between simple noncrossing trees and planar binary trees.*

Proof. By induction on the degree n . This is clearly true if $n = 1$, for \triangleleft and \blacktriangle .

If x is a planar binary tree of degree $n \geq 2$, then x can be written y/\swarrow , \swarrow/z or $y/\swarrow/z$ for some smaller planar binary trees y and z . By induction hypothesis, y and z are noncrossing trees with $N_{\blacktriangle}(y) = N_{\blacktriangle}(z) = 0$. Therefore x is also a noncrossing tree with $N_{\blacktriangle}(x) = 0$.

Conversely, if P is a noncrossing tree with $N_{\blacktriangle}(P) = 0$, then P must be based by Lemma 3.4. Therefore P can be written Q/\swarrow , \swarrow/R or $Q/\swarrow/R$ for some smaller noncrossing trees Q and R . In this case, one necessarily has $N_{\blacktriangle}(Q) = 0$ and $N_{\blacktriangle}(R) = 0$. Therefore each of Q and R is a single planar binary tree by induction. Hence the same is true for P . \blacksquare

Let us call a noncrossing tree P satisfying $N_{\blacktriangle}(P) = 0$ a **simple noncrossing tree**.

Note that pivots between simple noncrossing trees correspond to elementary moves between planar binary trees, *i.e.* edges in the Hasse diagram of the Tamari poset.

10 The $\#$ product and the $\#$ functor

On the direct sum Dend of all Abelian groups $\text{Dend}(n)$ for $n \geq 1$, there is an associative product $\#$, which can also be defined using jeu-de-taquin on planar binary trees. The author has learned about this product from Aval and Viennot, see [VA] and [Vie07] for the context in which they consider the $\#$ product.

Unlike the product $*$, the product $\#$ is graded with respect to the degree minus 1, namely it restricts to homogeneous maps $\text{Dend}(m) \otimes \text{Dend}(n) \rightarrow \text{Dend}(m+n-1)$.

In our context, the product $\#$ can be defined as follows:

$$x\#y = -\theta^{-1}(\theta(y) \circ_1 \theta(x)) = -\theta(\theta^{-1}(x) \circ_n \theta^{-1}(y)), \quad (27)$$

where x has degree n . The equality of the last two terms follows from the axiom (11) of anticyclic operads and from the result that $\tau = (-1)^n \theta^2$ on $\text{Dend}(n)$.

Proposition 10.1 *The $\#$ product is associative. The planar binary tree \vee is a unit for $\#$.*

Proof. The associativity follows from the associativity of \circ_1 , which in turn is a consequence of the axioms of operads. The fact that \vee is a unit follows from the fact that it is the unit of the Dendriform operad. ■

Proposition 10.2 *The reversal of planar binary trees is an anti-automorphism of the $\#$ product : $\overline{x\#y} = \overline{y}\#\overline{x}$.*

Proof. One uses the equivalent forms of the definition (27), the relation (18) between θ and reversal, and equation (2) relating reversal, \circ_1 and \circ_n . ■

Lemma 10.3 *One has*

$$(x * y)\#z = x * (y\#z), \quad (28)$$

and

$$(x\#y) * z = x\#(y * z). \quad (29)$$

Proof. By left-right symmetry, it is enough to prove the second equation. By definition (27), one has

$$x\#(y * z) = -\theta^{-1}(\theta(y * z) \circ_1 \theta(x)). \quad (30)$$

By (15), this becomes

$$\theta^{-1}((\theta(y)/\theta(z)) \circ_1 \theta(x)). \quad (31)$$

By (8), this is

$$\theta^{-1}((\theta(y) \circ_1 \theta(x))/\theta(z)), \quad (32)$$

which equals, by (16) and definition (27),

$$-\theta^{-1}(\theta(y) \circ_1 \theta(x)) * z = (x\#y) * z. \quad (33)$$

■

Lemma 10.4 *One has*

$$x\#(y\backslash z) = (x\#y)\backslash z. \quad (34)$$

and

$$(x/y)\#z = x/(y\#z). \quad (35)$$

Proof. By left-right symmetry, it is enough to prove the first equality. Using (9), one gets

$$(\theta(y) \circ_1 \theta(x)) * \theta(z) = (\theta(y) * \theta(z)) \circ_1 \theta(x). \quad (36)$$

This becomes, by definition (27) and (14),

$$-\theta(x\#y) * \theta(z) = -\theta(y\backslash z) \circ_1 \theta(x). \quad (37)$$

This is equivalent to

$$-\theta^{-1}(\theta(x\#y) * \theta(z)) = -\theta^{-1}(\theta(y\backslash z) \circ_1 \theta(x)). \quad (38)$$

Therefore we get, by definition (27) and (17),

$$(x\#y)\backslash z = x\#(y\backslash z). \quad (39)$$

■

The set of noncrossing trees is not closed under the $\#$ product, for instance $\blacktriangle\#\blacktriangleleft$ is not a noncrossing tree. One can nevertheless give a combinatorial description of the $\#$ product of noncrossing trees, when it is still a noncrossing tree. Let P and Q be noncrossing trees. If P or Q is not based, then they can be decomposed as $*$ products and one can use Lemma 10.3 to reduce to smaller $\#$ products. Let us now assume that P and Q are based. If $P = P' \vee$ or $Q = \vee \backslash Q'$, then $P\#Q$ can be described using Lemma 10.4 as P'/Q or $P \backslash Q'$, and is therefore a noncrossing tree.

Proposition 10.5 *The $\#$ product of two projective elements is a projective element.*

Proof. Let P and Q be projective elements. The proof is by induction on $\deg(P)$.

If P has degree 1, then $P = \vee$, therefore $P\#Q = Q$ and the result is true.

If P is based, then $P = \blacktriangleleft \circ_1 R$ for some projective element R by Proposition 6.1. Then $P\#Q = (\blacktriangleleft \circ_1 R)\#Q = (R/\vee)\#Q = R/Q$ by Lemma 10.4. This is a projective element by Prop. 4.4.

If P is not based, then it can be written $R * R'$ for some projective elements R and R' by Proposition 6.2. In this case, $P\#Q = (R * R')\#Q = R * (R'\#Q)$ by Lemma 10.3. By induction hypothesis, $R'\#Q$ is a projective element. Therefore $P\#Q$ is a projective element by Prop. 6.4. ■

Proposition 10.6 *The smallest subset of Dend containing $\{\blacktriangleleft, \blacktriangleright\}$ and stable under \circ_1 and $\#$ is the set of projective elements.*

Proof. Let us call this set \mathcal{A} . Then \mathcal{A} is contained in the set of projective elements, because \blacktriangleleft and \blacktriangleright are projective elements and \circ_1 and $\#$ preserves projective elements by Proposition 7.2 and Proposition 10.5.

Let us prove the reverse inclusion by induction on the degree n . This is true if $n = 2$. Let P be a projective element of degree at least 3. If P is based, then P can be written $\triangleleft \circ_1 Q$ for some projective element Q , by Proposition 6.1. By induction, Q is in \mathcal{A} , hence P is in \mathcal{A} .

If P is not based, then it can be written $Q * R$ with Q and R smaller projective elements. If $R = \vee$, then $P = (Q \# \vee) * \vee = Q \# \blacktriangle$ by Lemma 10.3. But Q is in \mathcal{A} by induction, therefore P too. If R has degree at least 2, then one has $P = Q * (\vee \# R) = (Q * \vee) \# R$ by Lemma 10.3. By induction, the projective elements $Q * \vee$ and R are in \mathcal{A} . Therefore P itself is in \mathcal{A} . ■

It follows from Prop. 10.5, by the same kind of argument as used before for \circ_1 , that one can define a functor $\#$ from the category of $\mathbb{Y}_m \times \mathbb{Y}_n$ modules to the category of \mathbb{Y}_{m+n-1} modules, that is a categorification of the $\#$ product.

This implies the following result.

Proposition 10.7 *The $\#$ product respects the Euler form of the Tamari posets: one has $E(x \# y) = E(x) \# E(y)$.*

The existence of the $\#$ functor also implies that every $\mathbb{Y}_m \times \mathbb{Y}_n$ -module is sent to a \mathbb{Y}_{m+n-1} module. At the level of the Grothendieck group, this implies that the $\#$ product is positive on positive elements.

Also one can deduce that the $\#$ product has the following property:

$$\left(\sum_{s \in \mathbb{Y}_m} s \right) \# \left(\sum_{t \in \mathbb{Y}_n} t \right) = \sum_{u \in \mathbb{Y}_{m+n-1}} u. \quad (40)$$

Together with positivity, this implies that the $\#$ product of two sums of planar binary trees without multiplicity is a sum of planar binary trees without multiplicity.

Proposition 10.8 *Let S be a subset of \mathbb{Y}_m and T be a subset of \mathbb{Y}_n . Then $S \# T$ is the sum over a subset of \mathbb{Y}_{m+n-1} .*

References

- [Ara] T. Araya. Exceptional Sequences over path algebras of type A_n and Non-crossing Spanning Trees. arXiv:0904.2831.
- [BW97] A. Björner and M. L. Wachs. Shellable nonpure complexes and posets. II. *Trans. Amer. Math. Soc.*, 349(10):3945–3975, 1997.
- [Cha07] F. Chapoton. Sur le nombre d’intervalles dans les treillis de Tamari. *Sém. Lothar. Combin.*, 55:Art. B55f, 18 pp. (electronic), 2005/07.
- [Cha05] F. Chapoton. On some anticyclic operads. *Algebr. Geom. Topol.*, 5:53–69 (electronic), 2005.
- [Cha07a] F. Chapoton. The anticyclic operad of moulds. *Int. Math. Res. Not. IMRN*, (20):Art. ID rnm078, 36, 2007.
- [Cha07b] F. Chapoton. On the Coxeter transformations for Tamari posets. *Canad. Math. Bull.*, 50(2):182–190, 2007.

- [Cha08] F. Chapoton. The categorified Diassociative cooperad. 2008.
- [HT72] S. Huang and D. Tamari. Problems of associativity: A simple proof for the lattice property of systems ordered by a semi-associative law. *J. Combinatorial Theory Ser. A*, 13:7–13, 1972.
- [Lod01] J.-L. Loday. Dialgebras. In *Dialgebras and related operads*, volume 1763 of *Lecture Notes in Math.*, pages 7–66. Springer, Berlin, 2001.
- [LR02] J.-L. Loday and M. O. Ronco. Order structure on the algebra of permutations and of planar binary trees. *J. Algebraic Combin.*, 15(3):253–270, 2002.
- [Pal93] J. M. Pallo. An algorithm to compute the Möbius function of the rotation lattice of binary trees. *RAIRO Inform. Théor. Appl.*, 27(4):341–348, 1993.
- [VA] X. Viennot and J.-C. Aval. The product of trees in the loday-ronco algebra through catalan alternative tableaux. article in preparation.
- [Vie07] X. Viennot. Catalan tableaux and the asymmetric exclusion process. In *Proc. FPSAC 07 (Formal Power Series and Algebraic Combinatorics)*, Tienjin, Chine, 2007. ArXiv: math-CO/0905.3081.